THE LEIBNIZ FORMULA FOR π

Consider the sum of the following geometric series, obtained by multiplying the previous term by $-x^2$...

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

The sum of geometric series, with common ratio r, is given by $S_\infty = \frac{1}{1-r}$, hence...

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \frac{1}{1+x^2}$$

Now integrate this equation with respect to x from 0 to 1...

$$\left[x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \cdots\right]_0^1 = \int_0^1 \frac{dx}{1+x^2}$$

The integral can be solved using the substitution $x = \tan\theta$...

$$\left[x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \cdots\right]_0^1 = \int_{x=0}^{x=1} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

Since $1 + \tan^2\theta = \sec^2\theta$ the integral reduces to...

$$\left[x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \cdots\right]_0^1 = \int_{x=0}^{x=1} d\theta = [\arctan(x)]_0^1$$

Evaluate this, using the fact that: $\tan\left({}^{\pi}/_{4}\right) = 1$ and $\tan(0) = 0$...

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$$

Finally, multiplying through by 4 gives...

$$\boxed{4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots = \pi}$$

This is the Leibniz formula for π.